

EMBEDDING L^1 IN L^1/H^1

BY

J. BOURGAIN

ABSTRACT. It is proved that L^1 is isomorphic to a subspace of L^1/H^1 . More precisely, there exists a diffuse σ -algebra $\mathfrak{S}$ on the circle such that the corresponding expectation $E: H^\infty \rightarrow L^\infty(\mathfrak{C})$ is onto. The method consists in studying certain martingales on the product $\Pi^\mathbb{N}$.

1. Introduction. Let us start by fixing some terminology. As usual, Π will denote the circle equipped with its Haar measure m , H_0^1 is the subspace of those $f \in L^1(\Pi)$ for which $\hat{f}(n) = 0$ for $n \leq 0$ and $q: L^1 \rightarrow L^1/H_0^1$ is the quotient map.

We are interested in the question whether or not there exists a linear embedding of the Banach space L^1 in the space L^1/H_0^1 . We briefly indicate some motivation for this problem. First, it was (and still remains) an open question if the three-space-property holds for L^1 -embedding, i.e. suppose X a Banach space, Y a subspace of X . Is it true that whenever L^1 embeds in X , it also has to embed in either Y or X/Y ?

The problem is also unsolved in the particular case $X = L^1$ and Y isomorphic to a dual space. It is not hard to show that an embedding of L^1 in X/Y is then equivalent to the existence of a subspace S of X , S isomorphic to L^1 so that the quotient map $X \rightarrow X/Y$ is an isomorphism when restricted to S .

In the special situation $X = L^1(\Pi)$ and $Y = H_0^1$, the answer was unknown for some time. There was hope that this may provide a counterexample in view of the following result, due to W. B. Johnson (see [9]).

PROPOSITION 1. *No complemented subspace of L^1/H_0^1 is isomorphic to L^1 .*

This is a consequence of the fact that any operator $T: L^1/H^1 \rightarrow L^1$ maps weakly compact sets onto norm compact sets. Let us sketch the argument.

Consider the identity map $I: L^\infty/H^\infty \rightarrow L^1/H^1$. Then $(TI)^*: L^\infty \rightarrow H^\infty \rightarrow H^1$ is integral and therefore nuclear (since H^1 satisfies the Radon-Nikodym property). Consequently, also TI is nuclear. Given now a weakly null sequence $(x_n)_{n=1,2,\dots}$ in L^1/H^1 , it follows from the lifting property (see [9] for instance) that $x_n = q(f_n)$ where $\{f_n; n = 1, 2, \dots\}$ is a relatively weakly compact set in $L^1(\Pi)$. Therefore, for each $\varepsilon > 0$, a truncation argument provides a bounded sequence (g_n) in L^∞ such that $\|f_n - g_n\|_1 < \varepsilon$ for each n . Thus

$$\|Tx_n - TI\tilde{g}_n\| \leq \|T\| \|x_n - I\tilde{g}_n\| < \varepsilon \|T\|.$$

Received by the editors April 22, 1982 and, in revised form, July 20, 1982.

1980 *Mathematics Subject Classification*. Primary 46E15, 42A99, 60G45.

©1983 American Mathematical Society
0002-9947/82/0000-0794/\$03.50

Because TI is nuclear, the set $\{TI(\tilde{g}_n); n = 1, 2, \dots\}$ is compact for each $\varepsilon > 0$. So we conclude that $\{Tx_n\}$ is compact, as announced.

Using Proposition 1, the following is proved in [2].

PROPOSITION 2. *There is no almost isometric embedding of the complex L^1 space in L^1/H^1 .*

Thus $d(S, L^1) > \gamma > 1$ for each subspace S of L^1/H_0^1 , where d is the Banach-Mazur distance (see [8, 9] for definitions). This observation allows us to define a natural distortion of L^1 , by taking

$$\|f\| = \|f\|_1 + \|q(f)\|, \quad f \in L^1(\Pi).$$

Say that an operator $T: X \rightarrow Y$ is a semiembedding provided T is one-one and maps the closed unit ball of X on a norm-closed subset of Y . It can be shown that a semiembedding $T: L^1 \rightarrow L^1$ has to fix an L^1 -copy (i.e. is an isomorphism when restricted to a subspace S of L^1 , S isomorphic to L^1). On the other hand, (see [3]):

PROPOSITION 3. *The restriction of the quotient map $q: L^1 \rightarrow L^1/H_0^1$ to the subspace $L^1_{\mathbf{R}}$ of real functions in $L^1(\Pi)$ is a semiembedding.*

No example is known of a semiembedding of L^1 in a Banach space X not containing L^1 .

Our purpose is to prove the existence of a natural embedding of L^1 in L^1/H_0^1 . There exists a diffuse σ -algebra $\mathfrak{S}$ on Π so that the restriction of q to the complex $L^1(\mathfrak{S})$ -space is an isomorphism. More precisely:

THEOREM. *There exists an increasing sequence (n_k) of positive integers, such that if $\mathfrak{S}$ is the σ -algebra on Π generated by the functions $\sigma_k(\theta) = \text{sign} \cos n_k \theta$, then the restriction of q to $L^1(\mathfrak{S})$ is an isomorphism. Consequently, for this σ -algebra $\mathfrak{S}$, the expectation operator $\mathbf{E}: H^\infty \rightarrow L^\infty(\mathfrak{S})$ is onto.*

The argument presented here is rather delicate. In order to give the reader an idea how it is organised, we briefly outline the proof. We have to introduce the σ -algebra $\mathfrak{S}$ such that the inequality

$$(*) \quad \|h - \mathbf{E}_{\mathfrak{S}}[h]\|_1 \geq \delta \|h\|_1$$

holds for each $h \in H_0^1$. But choosing the sequence (n_k) sufficiently lacunary, it is enough to verify $(*)$ for functions h with spectrum contained in a set of the form

$$E = \{\sum' \nu_k n_k; |\nu_k| \leq a_k \text{ for each } k\}$$

where (a_k) is a sequence of positive integers and (n_k) , (a_k) satisfy the transference property. Thus the n_k -frequencies can be replaced by independent variables. The space $H_0^1 \cap L_E^1$ identifies with a subspace of the space $\mathcal{K} \subset L^1(\Pi^{\mathbb{N}})$ of those functions $h = \sum h_k$ on $\Pi^{\mathbb{N}}$ such that each increment $h_k = h_k(x_1, \dots, x_k)$ is an H_0^1 -function in x_k . The required inequality now becomes

$$(**) \quad \|h - \mathbf{E}_{\mathfrak{F}}[h]\|_1 \geq \|h\|_1$$

for $h \in \mathcal{K}$, where $\mathfrak{F}$ is a natural diadic product σ -algebra on $\Pi^{\mathbb{N}}$ (generated by the functions $\sigma_k(x) = \text{sign} \cos x_k$).

This reduction of the problem is worked out in §4. Its purpose is to approach the problem with martingale techniques. The martingale prerequisites are given in §2. To obtain (**) we first prove L^1 -estimations for certain square functions related to h (see Lemma 4). These are derived using a "step-by-step" method (explained at the beginning of §5) and an examination of what happens at each increment. More precisely, we have to consider at this point functions of the form $a + h - b\sigma$, where a, b are scalars, $h \in H_0^1$ and $\sigma = \text{sign cos}$.

Minorations of the L^1 -norm of such expressions are given in Propositions 8 and 9 below. It is only at this place that some complex function theory will be involved.

2. Martingale preliminaries. Let $(\mathcal{F}_k)_{k=0,1,2,\dots}$ be an increasing sequence of σ -algebras on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ assuming $\mathcal{F} = \bigvee_{k=1}^{\infty} \mathcal{F}_k$. Denote by $\mathbf{E}_k$ the expectation with respect to $\mathcal{F}_k$. For $f \in L^1(\mathcal{F})$ let

$$f^* = \sup_k |\mathbf{E}_k[f]| \quad \text{and} \quad S(f) = \left[|\mathbf{E}_0[f]|^2 + \sum_{k=1}^{\infty} |\mathbf{E}_k[f] - \mathbf{E}_{k-1}[f]|^2 \right]^{1/2}.$$

We will use the notation C to indicate a numerical constant. Let us recall the following result, due to D. Davis (see [7]).

PROPOSITION 4. $C^{-1} \|S(f)\|_1 \leq \|f^*\|_1 \leq C \|S(f)\|_1$.

The next inequality is probably known, but we include its proof here for the sake of completeness.

PROPOSITION 5. *Let (v_k) be an adapted sequence of functions; thus v_k is $\mathcal{F}_k$ -measurable for each k . Then*

$$\left\| \left[\sum |\mathbf{E}_{k-1}[|v_k|]|^2 \right]^{1/2} \right\|_1 \leq C \left\| \left[\sum |v_k|^2 \right]^{1/2} \right\|_1.$$

PROOF. It is no restriction to assume the $\mathcal{F}_k$ finite algebras. Moreover, since one may always tensor the v_k against a Rademacher sequence, we can assume $\mathbf{E}_{k-1}[v_k] = 0$ and thus (v_k) is an adapted martingale difference sequence. Since, then

$$\left\| \left[\sum |v_k|^2 \right]^{1/2} \right\|_1 = \left\| \sum v_k \right\|_{H^1(\mathcal{F}_k)},$$

it follows from the atomic decomposition property for H^1 -functions (see for instance [7, Chapter I]) and convexity, that we may take for $\sum v_k$ a function of the form (for some positive integer j)

$$a = \frac{1}{|A|} (\varphi - \mathbf{E}_{j-1}[\varphi])$$

where A is an $\mathcal{F}_j$ -atom, $\text{supp } \varphi \subset A$ and $\|\varphi\|_{\infty} \leq 1$. In this case

$$\begin{aligned} v_k &= \mathbf{E}_k[a] - \mathbf{E}_{k-1}[a] = 0 \quad \text{for } k < j, \\ &= \frac{1}{|A|} (\mathbf{E}_k[\varphi] - \mathbf{E}_{k-1}[\varphi]) \quad \text{for } k \geq j. \end{aligned}$$

Also, $\mathbf{E}_k[\varphi]$ is supported by A for $k \geq j$ and hence v_k for $k > j$. Thus the left side in Proposition 5 is dominated by

$$\begin{aligned} \|v_j\|_1 + \left\| \left(\sum_{k>j} \mathbf{E}_{k-1}[|v_k|^2] \right)^{1/2} \right\|_1 \\ \leq 2 + \int_A \left(\sum_{k>j} \mathbf{E}_{k-1}[|v_k|^2] \right)^{1/2} \quad (\text{by Cauchy-Schwarz}) \\ \leq 2 + |A|^{1/2} \left(\int \sum_{k>j} |v_k|^2 \right)^{1/2} \\ \leq 2 + |A|^{1/2} \|a\|_2 \leq 3, \end{aligned}$$

proving the result.

PROPOSITION 6. *For $f \in H^1(\mathcal{G}_k)$, one has an inequality*

$$\left(\sum \|(\mathbf{E}_k - \mathbf{E}_{k-1})[f]\|_1^2 \right)^{1/2} \leq C \|f\|_1^{1/2} \|f\|_{H^1}^{1/2}.$$

To prove this, we will first deal with the special case of the Rademacher projection on the Cantor group (in fact, only this will be used later on).

PROPOSITION 7. *If $D = \{1, -1\}^{\mathbb{N}}$ is the Cantor-group and $f \in H^1(D)$, then*

$$\left(\sum |\hat{f}(k)|^2 \right)^{1/2} \leq C \|f\|_1^{1/2} \|f\|_{H^1}^{1/2}$$

where $\hat{f}(k) = \int f(\varepsilon) \varepsilon_k$.

PROOF. We will use the theorem of [6] on the BMO-distance of a BMO-function to L^∞ (in the dyadic setting). The result asserts, in particular, that for $\varphi \in \text{BMO}(D)$, $\text{dist}_{\text{BMO}}(\varphi, L^\infty) = 0$ and $\varepsilon > 0$, there exists a decomposition $\varphi = \alpha + \beta$ such that

$$\|\alpha\|_{\text{BMO}} \leq C_1 \varepsilon \quad \text{and} \quad \|\beta\|_\infty \leq C_2 \max(\varepsilon, \lambda_0(\varepsilon))$$

where $\lambda_0 = \lambda_0(\varepsilon)$ has to satisfy

$$\sup_I \frac{1}{|I|} \left| \{x \in I, |\varphi(x) - \varphi_I| > \lambda\} \right| \leq e^{-\lambda/\varepsilon}$$

whenever $\lambda > \lambda_0$ ($\varphi_I = |I|^{-1} \int_I \varphi$).

Now take $\varphi = \sum a_k \varepsilon_k$ with $\sum |a_k|^2 = 1$. It follows from the distribution property of Rademacher that for each dyadic interval I ,

$$\left| \{ \alpha \in I; |\varphi(x) - \varphi_I| > \lambda \} \right| \leq C e^{-c\lambda^2} |I|,$$

for numerical constants $c > 0$, $C < \infty$. Hence $\text{dist}_{\text{BMO}}(\varphi, L^\infty) = 0$ and $\lambda_0(\varepsilon) \sim 1/\varepsilon$.

Decomposing $\varphi = \alpha + \beta$ as above, we get

$$|\langle f, \varphi \rangle| \leq |\langle f, \alpha \rangle| + |\langle f, \beta \rangle| \leq C_1 \varepsilon \|f\|_{H^1} + C_2 \frac{1}{\varepsilon} \|f\|_1.$$

Taking supremum over φ and choosing $\varepsilon = \|f\|_1^{1/2} \|f\|_{H^1}^{-1/2}$, the inequality follows.

PROOF OF PROPOSITION 6. Assume f real and estimate

$$\left(\sum_{k=1}^K \|(\mathbf{E}_k - \mathbf{E}_{k-1})[f]\|_1^2 \right)^{1/2}.$$

Define for each k ,

$$\sigma_k = \text{sign } \Delta f_k \quad \text{and} \quad b_k = \frac{1}{2}(\sigma_k - \mathbf{E}_{k-1}[\sigma_k]).$$

Then

$$\|f\|_1 \geq \iint |f| \prod_{k=1}^K (1 + \varepsilon_k b_k) d\varepsilon \mathbf{P}(d\omega) \geq \frac{1}{2} \int \left| \sum_{k=1}^K \varepsilon_k \Phi_k(\varepsilon) \right| d\varepsilon$$

where

$$\Phi_k(\varepsilon) = \int \prod_{j=1}^{k-1} (1 + \varepsilon_j b_j) |\Delta f_k| d\omega.$$

Application of Proposition 7 to the function $\sum \varepsilon_k \Phi_k(\varepsilon)$ then gives

$$\begin{aligned} \left(\sum_{k=1}^K \|\Delta f_k\|_1^2 \right)^{1/2} &\leq C \|f\|_1^{-1/2} \left[\int \left(\sum |\Phi_k(\varepsilon)|^2 \right)^{1/2} d\varepsilon \right]^{1/2} \\ &\leq C \|f\|_1^{1/2} \left[\iint S(f) \prod (1 + \varepsilon_j b_j) d\omega d\varepsilon \right]^{1/2} \\ &= C \|f\|_1^{1/2} \|f\|_{H^1}^{1/2} \end{aligned}$$

as announced.

REMARK. The author is grateful to P. W. Jones for outlining a more explicit procedure to obtain the decomposition used in the proof of Proposition 7.

3. Some inequalities involving H_0^1 -functions. The purpose of this section is to prove the following results.

PROPOSITION 8. For $a \in \mathbf{C}$ and $h \in H_0^1$, one has

$$\|a + h\|_1 \geq \left\| \left(|a|^2 + \delta^2 |h|^2 \right)^{1/2} \right\|_1$$

where $\delta > 0$ is a fixed constant.

PROPOSITION 9. There exists $\delta > 0$ such that for $a \in \mathbf{C}$, $b \in \mathbf{C}$ and $h \in H_0^1$,

$$(i) \quad \|a + h - b\sigma\|_1 \geq \left\{ |a|^2 + \delta^2 \left[\frac{\text{Re}(\overline{\langle h, \sigma \rangle} (\langle h, \sigma \rangle - b))}{|\langle h, \sigma \rangle| + |b|} \right]^2 \right\}^{1/2},$$

$$(ii) \quad \|a + h - \langle h, \sigma \rangle \sigma\|_1 \geq \left\| \left\{ |a|^2 + \delta^2 |h_e - \langle h, \sigma \rangle \sigma|^2 \right\}^{1/2} \right\|_1$$

where $\sigma = \text{sign } \cos$ and $h_e(\theta) = \sum_{n=1}^{\infty} \hat{h}(n) \cos n\theta$.

It is clear that it suffices to prove Propositions 8 and 9, with $a = 1$.

PROOF OF PROPOSITION 8. Factoring $1 + h$ gives $1 + h = (1 + g_1)(1 + g_2)$ where $g_1, g_2 \in H_0^2$ and

$$\|1 + h\|_1 = (1 + \|g_1\|_2^2)^{1/2} (1 + \|g_2\|_2^2)^{1/2}.$$

Since $|h| \leq |g_1| + |g_2| + |g_1| |g_2|$ the result follows from the majorations

$$\left\| (1 + |g_i|^2)^{1/2} \right\|_1 \leq \left\| (1 + |g_i|^2)^{1/2} \right\|_2 = (1 + \|g_i\|_2^2)^{1/2} \leq \|1 + h\|_1 \quad (i = 1, 2)$$

and

$$\left\| (1 + |g_1|^2 |g_2|^2)^{1/2} \right\|_1 \leq 1 + \|g_1 g_2\|_1 \leq 1 + \|g_1\|_2 \|g_2\|_2 \leq \|1 + h\|_1.$$

Also to obtain Proposition 9, we will use the L^2 -theory. Our argument here is, however, more complicated. This is the only point where explicit constructions of H^∞ -functions appear.

LEMMA 1. *Given a measurable subset A of Π , there exist H^∞ -functions φ and ψ satisfying the following conditions:*

- (i) $|\varphi| + |\psi| \leq 1$,
- (ii) $\operatorname{Re} \psi$ is an even function on Π ,
- (iii) $|\varphi - 1/8| < 1/100$ on the set A ,
- (iv) $\|\varphi\|_1 \leq C|A|$,
- (v) $\|\operatorname{Re} \psi - 1\|_1 \leq C|A|$.

PROOF. Fix some (large) $M > 0$ and define the following H^∞ -functions:

$$\begin{aligned} \tau(z) &= -M \int_A \frac{e^{i\theta} + z}{e^{i\theta} - z} m(d\theta), \quad \varphi = \frac{1}{8} (1 - e^\tau)^2, \\ \psi(z) &= \exp \left\{ \int_\Pi \log(1 - \alpha(\theta)) \frac{e^{i\theta} + z}{e^{i\theta} - z} m(d\theta) \right\} \end{aligned}$$

where $\alpha(\theta) = |\varphi(e^{i\theta})| \vee |\varphi(e^{-i\theta})|$.

Notice that this makes sense, because e^τ has boundary value $e^{-M(\chi_A + i\mathcal{H}(\chi_A))}$ ($\mathcal{H}$ = Hilbert-transform) and therefore $\|\alpha\|_\infty \leq \frac{1}{2}$.

(i) is obvious. On Π , we have $\operatorname{Re} \psi = (1 - \alpha) \cos \mathcal{H}(\log(1 - \alpha))$ and thus an even function. Since $|\varphi - \frac{1}{8}| \leq \frac{3}{8} |e^\tau|$ and thus $|\varphi - \frac{1}{8}| < e^{-M}$ on A (iii) holds for M large enough. Because on Π

$$8|\varphi| \leq \chi_A + M^2 |\mathcal{H}(\chi_A)|^2,$$

(iv) follows. Finally,

$$|1 - \operatorname{Re} \psi| \leq |\alpha| + \frac{1}{2} |\mathcal{H}(\log(1 - \alpha))|^2, \quad \|1 - \operatorname{Re} \psi\|_1 \leq 4\|\varphi\|_1$$

and hence (v).

We refer the reader to [4, Proposition 1.6] for the following Marcinkiewicz type decomposition.

LEMMA 2. *There is a constant $C < \infty$ such that for given $h \in H_0^1$ and $\lambda > 0$, there exists $h_\lambda \in H_0^\infty$ satisfying:*

- (i) $|h_\lambda| \leq C|h|$,
- (ii) $\|h_\lambda\|_\infty \leq C\lambda$,
- (iii) $\|h - h_\lambda\|_1 \leq C \int_{|h| > \lambda} |h|$.

Let h be as in Proposition 9. For $\lambda > 0$, define $A_\lambda = [|h| > \lambda]$. Application of Lemma 1 to the set A_λ provides H^∞ -functions $\varphi_\lambda, \psi_\lambda$. We are now ready to prove

LEMMA 3. $\|1 + h - b\sigma\|_1 \geq 1 + c \int_{A_\lambda} |h| + c\lambda^{-2} \|\operatorname{Im}(h_\lambda - b\sigma)\|_2^2$ if $\lambda > K$ and $|b| < \lambda/K$ ($c > 0$ and $K < \infty$ being numerical constants).

PROOF. First, since $1 - b\sigma$ is even and $\operatorname{Im} \psi_\lambda$ odd, we find

$$\begin{aligned} \|1 + h - b\sigma\|_1 &\geq \|(1 + h - b\sigma)\varphi_\lambda\|_1 + \left| \int (1 - b\sigma)\psi_\lambda \right| \\ &\geq \frac{1}{9} \int_{A_\lambda} |h| - (1 + |b|) + \left| \int (1 - b\sigma) \operatorname{Re} \psi_\lambda \right| \\ &\geq \frac{1}{9} \int_{A_\lambda} |h| - \frac{1}{9} (1 + |b|) |A_\lambda| + 1 - (1 + |b|) \|1 - \operatorname{Re} \psi_\lambda\|_1 \\ &\geq \frac{1}{9} \int_{A_\lambda} |h| - C(1 + |b|) |A_\lambda| + 1 \end{aligned}$$

for some constant C . Thus, choosing K large enough, we get

$$(*) \quad \|1 + h - b\sigma\|_1 \geq 1 + \frac{1}{10} \int_{A_\lambda} |h|.$$

Fix some small constant $\delta > 0$. Since we always have

$$\|1 + af\|_1 \leq \|1 + f\|_1 \quad \text{for } 0 \leq a \leq 1 \text{ and } f \text{ of mean } 0,$$

it follows that

$$\|1 + h - b\sigma\|_1 \geq \|1 + \delta\lambda^{-1}(h - b\sigma)\|_1 \geq \|1 + \delta\lambda^{-1}(h_\lambda - b\sigma)\|_1 - \delta\lambda^{-1}\|h - h_\lambda\|_1.$$

Because $\delta\lambda^{-1}|h_\lambda - b\sigma| \leq 1$ the inequality

$$(1 + t)^{1/2} \geq 1 + t/3 \quad \text{for } 0 \leq t \leq 1$$

yields

$$|1 + \delta\lambda^{-1}(h_\lambda - b\sigma)| \geq [1 + \delta\lambda^{-1} \operatorname{Re}(h_\lambda - b\sigma)] \left[1 + \frac{1}{12} \delta^2 \lambda^{-2} (\operatorname{Im}(h_\lambda - b\sigma))^2 \right].$$

Therefore, also

$$(**) \quad \|1 + h - b\sigma\|_1 \geq 1 + \frac{1}{20} \delta^2 \lambda^{-2} \int \operatorname{Im}^2(h_\lambda - b\sigma) - c\delta\lambda^{-1} \int_{A_\lambda} |h|.$$

The required minoration clearly follows combining (*) and (**).

PROOF OF PROPOSITION 9. *First*

$$\|1 + h - b\sigma\|_1 \geq d(b\sigma, H^1) \geq \frac{|b|}{2\pi} \left| \int_{-\pi}^{\pi} \sigma(\theta) e^{i\theta} d\theta \right| = \frac{2}{\pi} |b|$$

and hence, also,

$$\|1 + h - b\sigma\|_1 \geq \frac{1}{3}\|1 + h\|_1 \geq \frac{1}{3}\|h\|_1.$$

Notice that the right member of (i), (ii) is bounded by $1 + 2\delta\|h\|_1$. Since now $\|1 + h - b\sigma\|_1 \geq \frac{1}{6}\|h\|_1 + \frac{1}{6}|b|$, it follows that (i) (resp. (ii)) are satisfied for $|b| \geq 6$ (resp. $|\langle h, \sigma \rangle| \geq 6$). Hence, we may assume $|b| \leq M$ in (i), $|\langle h, \sigma \rangle| \leq M$ in (ii) where M is some numerical constant.

Fix a constant $\lambda > KM$ and put $k = h_\lambda$ for simplicity. Using again Lemma 2(iii), the right member of (i) can be majorized by

$$\left[1 + 2\delta^2(|\operatorname{Re}\langle h, \sigma \rangle|^2 + |\operatorname{Im}(\langle h, \sigma \rangle - b)|^2)\right]^{1/2} \\ \leq \left[1 + 2\delta^2(|\operatorname{Re}\langle k, \sigma \rangle|^2 + |\operatorname{Im}(\langle k, \sigma \rangle - b)|^2)\right]^{1/2} + 2\delta C \int_{A_\lambda} |h|.$$

Taking Lemma 3 into account, we see that it suffices to check the inequality

$$|\operatorname{Re}\langle k, \sigma \rangle|^2 + |\operatorname{Im}(\langle k, \sigma \rangle - b)|^2 \leq \|\operatorname{Im}(k - b\sigma)\|_2^2$$

which is straightforward:

$$\|\operatorname{Im}(k - b\sigma)\|_2^2 = \frac{1}{2} \sum_{n>0} |\operatorname{Im} \hat{k}(n) - 2 \operatorname{Im} b \hat{\sigma}(n)|^2 + \frac{1}{2} \sum_{n>0} |\operatorname{Re} \hat{k}(n)|^2$$

while

$$|\operatorname{Re}\langle k, \sigma \rangle| \leq \sum_{n>0} |\operatorname{Re} \hat{k}(n)| \hat{\sigma}(n) \leq \frac{1}{\sqrt{2}} \left(\sum |\operatorname{Re} \hat{k}(n)|^2 \right)^{1/2}, \\ |\operatorname{Im}(\langle k, \sigma \rangle - b)| \leq \sum_{n>0} |\operatorname{Im} \hat{k}(n) - 2 \operatorname{Im} b \hat{\sigma}(n)| \hat{\sigma}(n) \\ \leq \frac{1}{\sqrt{2}} \left(\sum_{n>0} |\operatorname{Im} \hat{k}(n) - 2 \operatorname{Im} b \hat{\sigma}(n)|^2 \right)^{1/2}.$$

For the right member of (ii), a similar reasoning reduces the question to the verification of

$$\int |k_e - \langle k, \sigma \rangle \sigma|^2 \leq \|\operatorname{Im}(k - b\sigma)\|_2^2,$$

which the reader will easily do.

4. Reduction of the problem. In this section, we will reduce the problem of proving that certain elements of $L^1(\Pi)$ normed by the quotient norm L^1/H^1 to the verification of an inequality for certain functions in $L^1(\Pi^N)$, where $\Pi^N = \Pi \times \Pi \times \dots$ is the product group. Denote by $\mathbf{E}_k$ ($k = 1, 2, \dots$) the expectation with respect to the k first variables $(x_1, x_2, \dots, x_k)$, where $x = (x_1, x_2, \dots)$ is the product variable.

We consider the subspace $\mathcal{H}$ of $L^1(\Pi^N)$ of those functions h such that for each k the difference $\mathbf{E}_k[h] - \mathbf{E}_{k-1}[h]$ is an H_0^1 -function with respect to x_k . Thus h is of the form

$$h = \sum h_k \quad \text{where } h_k = \sum_{n>0} \hat{h}_k(n) e^{in x_k}$$

and the $\hat{h}_k(n)$ are functions of $x_1, \dots, x_{k-1}$.

Again let $\sigma = \text{sign cos}$ and $\sigma_k(x) = \sigma(x_k)$ for each k . Let $\mathfrak{F}$ be the σ -algebra on $\Pi^{\mathbb{N}}$ generated by the σ_k . In the next section, we show the following

PROPOSITION 10. *There is a constant $c > 0$ s.t. $\|h - \mathbf{E}_{\mathfrak{F}}[h]\|_1 \geq c\|h\|_1$ for all $h \in \mathfrak{H}$.*

This fact obviously implies

COROLLARY 11. $\inf_{h \in \mathfrak{H}} \|f - h\|_1 \geq c'\|f\|_1$ for all $f \in L^1(\mathfrak{F})$.

For a, n positive integers, $\mathfrak{F}_a$ will be the Fejér kernel

$$F_a(\theta) = \sum_{|j| \leq a} \frac{a+1-|j|}{a+1} e^{ij\theta}$$

and $F_{a,n}(\theta) = F_a(n\theta)$.

We consider sequences of positive integers (n_k) , (a_k) satisfying the following conditions: ($\mathfrak{S}$ denotes again the σ -algebra on Π generated by the functions $\sigma(n_k\theta)$.)

(i) The transference property, i.e. let $E = \{\sum' \nu_k n_k; (\nu_k) \in F\}$ where F is the subset $\{(\nu_k), |\nu_k| \leq a_k\}$ of the dual group of $\Pi^{\mathbb{N}}$. Then the operator

$$T: L_E^1(\Pi) \rightarrow L_F^1(\Pi^n), \quad T(f)(x) = \sum_{(\nu_k) \in F} \hat{f}\left(\sum \nu_k n_k\right) e^{i(\sum \nu_k x_k)}$$

satisfies

$$\frac{1}{2}\|f\|_1 \leq \|T(f)\|_1 \leq 2\|f\|_1.$$

Moreover, $T(f) \in \mathfrak{H}$ for $f \in L_E^1 \cap H_0^1$.

(ii) Defining for each k ,

$$\xi_k = \sigma * F_{a_k}, \quad K = \prod_k F_{a_k, n_k},$$

$$R(\theta, \psi) = \prod [1 + \xi_k(n_k\theta)\sigma(n_k\psi)],$$

one has

$$(\alpha) \sum \|\xi_k - \sigma\|_1 \leq \varepsilon,$$

$$(\beta) \|K\|_1 = 1.$$

For $f \in L^1(\mathfrak{S})$,

$$(\gamma) \|f - f * K\|_1 < \varepsilon \|f\|_1,$$

(δ) $\|f - R(f)\|_1 < \varepsilon \|f\|_1$ where $R(f)(\theta) = \int f(\psi) R(\theta, \psi) m(d\psi)$ (where $\varepsilon > 0$ is a small constant).

The reader will easily convince himself that the realisation of these conditions is straightforward. Details on the transference property can be found in [1].

Let us now show that the sequence (n_k) satisfies the Theorem. Thus, fix $f \in L^1(\mathfrak{S})$ and $h \in H_0^1$. We get, by (ii),

$$\|f - h\|_1 \geq \|f * K - h * K\|_1 \geq \|R(f) - h * K\|_1 - 2\varepsilon \|f\|_1.$$

Notice that $R(f) \in L_E^1$. By (i),

$$\|R(f) - h * K\|_1 \geq \frac{1}{2} \|T(R(f)) - h_1\|_1$$

where $h_1 = T(h * K) \in \mathfrak{H}$.

Now

$$T(R(f))(x) = \int f(\psi) \prod [1 + \xi_k(x_k) \sigma(n_k \psi)] m(d\psi).$$

By (ii)(α), we see that for any (± 1) -sequence (τ_k)

$$\left\| \bigotimes (1 + \tau_k \xi_k) - \prod (1 + \tau_k \sigma_k) \right\|_1 < \varepsilon$$

implying that

$$\|T(R(f)) - f_1\| \leq 2\varepsilon \|f\| \quad \text{where } f_1 = \mathbf{E}[T(R(f))].$$

It follows then from Corollary 11 that

$$\begin{aligned} \|f - h\|_1 &\geq \frac{1}{2} \|f_1 - h_1\|_1 - 3\varepsilon \|f\|_1 \geq \frac{c}{2} \|f_1\|_1 - 3\varepsilon \|f\|_1 \\ &\geq \frac{c}{2} \|T(R(f))\|_1 - 4\varepsilon \|f\|_1 \geq \frac{c}{4} \|f\|_1 - 5\varepsilon \|f\|_1 \geq c' \|f\|_1 \end{aligned}$$

taking $\varepsilon > 0$ small enough.

5. Proof of the Theorem. It remains to prove Proposition 10. So fix $h = \sum h_k \in \mathcal{H}$ where

$$h_k = \sum_{n>0} \hat{h}_k(n) (x_1, \dots, x_{k-1}) e^{inx_k}.$$

We also define

$$\begin{aligned} [h_k]_e &= \sum \hat{h}_k(n) \cos nx_k, \\ [h_k]_0 &= \sum \hat{h}_k(n) \sin nx_k, \\ \langle h_k, \sigma_k \rangle &= \sum \hat{h}_k(n) \hat{\sigma}(\eta) \end{aligned}$$

(which is thus a function of $x_1, \dots, x_{k-1}$). If $f = \mathbf{E}_{\mathcal{G}}[h]$, then $f = \sum b_k \cdot \sigma_k$, where $b_k = b_k(x_1, \dots, x_{k-1}) = \mathbf{E}_{\mathcal{G}}[\langle h_k, \sigma_k \rangle]$.

Using E. Stein's theorem on H^1 -multipliers (see [11]), it is easily seen that $\|h\|_1 \sim \|S(h)\|_1$ (S = square function with respect to the natural decomposition).

We give a direct proof of this fact, based on Proposition 8.

Fix $1 > \varepsilon > 0$ and a positive sequence $(s_k)_{k=1,2,\dots}$ in $L^\infty(\Pi^{\mathbb{N}})$ satisfying $\|(\sum s_k^2)^{1/2}\|_\infty \leq \varepsilon$. Fixing a positive integer K , we get, using Proposition 8,

$$\begin{aligned} \|\mathbf{E}_K[h]\|_1 &= \|\mathbf{E}_{K-1}[h] + h_K\|_1 \\ &\geq \left\| \left(|\mathbf{E}_{K-1}[h]|^2 + \delta^2 |h_K|^2 \right)^{1/2} \right\|_1 \\ &\geq \left\| |\mathbf{E}_{K-1}[h]| (1 - s_K^2)^{1/2} \right\|_1 + \delta \| |h_K| s_K \|_1 \\ &\geq \|\mathbf{E}_{K-1}[h]\|_1 + \delta \| |h_K| s_K \|_1 - \|\mathbf{E}_{K-1}[h] s_K^2\|_1. \end{aligned}$$

Iterating,

$$\begin{aligned} \|h\|_1 &\geq \delta \sum \| |h_k| s_k \|_1 - \sum \|\mathbf{E}_{k-1}[h] s_k^2\|_1 \\ &\geq \delta \sum \| |h_k| s_k \|_1 - \varepsilon^2 \left\| \max_k |\mathbf{E}_k[h]| \right\|_1. \end{aligned}$$

Taking supremum over the sequences (s_k) , it follows that

$$\|h\|_1 \geq \delta \varepsilon \|S(h)\|_1 - \varepsilon^2 \left\| \max_k |\mathbf{E}_k[h]| \right\|_1$$

and choosing

$$\varepsilon^2 = \frac{\|h\|_1}{\left\| \max_k |\mathbf{E}_k[h]| \right\|_1},$$

we get

$$\|S(h)\|_1 \leq \delta^{-1} \|h\|_1^{1/2} \left\| \max_k |\mathbf{E}_k[h]| \right\|_1^{1/2}.$$

Hence, by Proposition 4, $\|S(h)\|_1 \leq \delta^{-2} \|h\|_1$ as required.

Before continuing, notice that since $\mathfrak{F}$ -expectation is a contraction, $\|S(f)\|_1 \leq \|S(h)\|_1$. Since for each k , $|\langle h_k, \sigma_k \rangle| \leq \mathbf{E}_{k-1}[|h_k|]$, application of Proposition 5 yields

$$\left\| \left(\sum_k |\langle h_k, \sigma_k \rangle|^2 \right)^{1/2} \right\|_1 \leq C \|h\|_1.$$

If we now apply the previous procedure using Proposition 9, the following inequalities are derived.

LEMMA 4.

$$(I) \quad \left\| \left\{ \sum_k \left| \frac{\operatorname{Re} \overline{\langle h_k, \sigma_k \rangle} (\langle h_k, \sigma_k \rangle - b_k)}{|\langle h_k, \sigma_k \rangle| + |b_k|} \right|^2 \right\}^{1/2} \right\|_1 \leq C \|h - f\|_1^{1/2} \|h\|_1^{1/2},$$

$$(II) \quad \left\| \left\{ \sum_k |[h_k]_8 - \langle h_k, \sigma_k \rangle \sigma_k|^2 \right\}^{1/2} \right\|_1 \leq C \|h - \sum \langle h_k, \sigma_k \rangle \sigma_k\|_1^{1/2} \|h\|_1^{1/2}.$$

PROOF. Let us show how (I) follows from Proposition 9(i). The argument for (II) is analogous. Fix $0 < \varepsilon < 1$ and a sequence $(s_k)_{k=1,2,\dots}$ of positive L^∞ -functions on Π^N satisfying $\|(\sum s_k^2)^{1/2}\|_\infty \leq \varepsilon$. Fix an integer k and apply Proposition 9(i) in the variable x_k . We get

$$\begin{aligned} \|\mathbf{E}_k[h - f]\|_1 &= \|\mathbf{E}_{k-1}[h - f] + h_k - b_k \sigma_k\|_1 \\ &\geq \left\| \left\{ |\mathbf{E}_{k-1}[h - f]|^2 + \delta^2 \left[\frac{\operatorname{Re} \overline{\langle h_k, \sigma_k \rangle} (\langle h_k, \sigma_k \rangle - b_k)}{|\langle h_k, \sigma_k \rangle| + |b_k|} \right]^2 \right\}^{1/2} \right\|_1 \\ &\geq \|\mathbf{E}_{k-1}[h - f]\|_1 + \delta \left\| \frac{\operatorname{Re} \overline{\langle h_k, \sigma_k \rangle} (\langle h_k, \sigma_k \rangle - b_k)}{|\langle h_k, \sigma_k \rangle| + |b_k|} s_k \right\|_1 \\ &\quad - \|\mathbf{E}_{k-1}[h - f] s_k^2\|_1. \end{aligned}$$

Iterating and using the same considerations as in the beginning of this section it follows that the left member of (I) is dominated by

$$\delta^{-1} \varepsilon^{-1} \|h - f\|_1 + \text{const. } \varepsilon \|S(h - f)\|_1,$$

and hence, choosing ε appropriately, by the right member of (I). We first make use of (I) to show

LEMMA 5. $\|\sum |\langle h_k, \sigma_k \rangle - b_k|^2\|_1^{1/2} \leq C \|h - f\|_1^{1/4} \|h\|_1^{3/4}$.

PROOF. Write

$$2 \frac{\operatorname{Re} \overline{\langle h_k, \sigma_k \rangle} (\langle h_k, \sigma_k \rangle - b_k)}{|\langle h_k, \sigma_k \rangle| + |b_k|} = \xi_k - |b_k|$$

where

$$\xi_k = \frac{|\langle h_k, \sigma_k \rangle - b_k|^2}{|\langle h_k, \sigma_k \rangle| + |b_k|} + |\langle h_k, \sigma_k \rangle|.$$

By the triangle inequality, the left side of (I) dominates

$$\left\| \left(\sum |\xi_k|^2 \right)^{1/2} \right\|_1 - \left\| \left(\sum |b_k|^2 \right)^{1/2} \right\|_1.$$

Also, since $b_k = \mathbf{E}[\langle h_k, \sigma_k \rangle]$,

$$\left\| \left(\sum |b_k|^2 \right)^{1/2} \right\|_1 \leq \left\| \left(\sum |\langle h_k, \sigma_k \rangle|^2 \right)^{1/2} \right\|_1.$$

Write

$$\begin{aligned} & \left[\sum (\xi_k^2 - |\langle h_k, \sigma_k \rangle|^2) \right]^{1/2} \\ &= \left[\left(\sum \xi_k^2 \right)^{1/2} + \left(\sum |\langle h_k, \sigma_k \rangle|^2 \right)^{1/2} \right]^{1/2} \left[\left(\sum \xi_k^2 \right)^{1/2} - \left(\sum |\langle h_k, \sigma_k \rangle|^2 \right)^{1/2} \right]^{1/2} \end{aligned}$$

and apply Cauchy-Schwarz. From (I) and previous observations

$$\left\| \left[\sum (\xi_k^2 - |\langle h_k, \sigma_k \rangle|^2) \right]^{1/2} \right\|_1 \leq C \|h\|_1^{1/2} \|h - f\|_1^{1/4} \|h\|_1^{1/4} = C \|h - f\|_1^{1/4} \|h\|_1^{3/4}.$$

Since for each k ,

$$\xi_k^2 - |\langle h_k, \sigma_k \rangle|^2 = (\xi_k + |\langle h_k, \sigma_k \rangle|) \frac{|\langle h_k, \sigma_k \rangle - b_k|^2}{|\langle h_k, \sigma_k \rangle| + |b_k|} \geq C |\langle h_k, \sigma_k \rangle - b_k|^2.$$

Lemma 5 is proved.

The left side of Lemma 5 dominates $\|f - \sum \langle h_k, \sigma_k \rangle \sigma_k\|_1$.

LEMMA 6. $\|\sum [h_k]_0\|_1$ and $\|\sum [h_k]_e - b_k \sigma_k\|_1^{1/2} \leq C \|h - f\|_1^{1/8} \|h\|_1^{7/8}$.

PROOF. Since $\sum [h_k]_0 = h - \sum [h_k]_e$, the first inequality is a consequence of the second. Write

$$\begin{aligned} \left\| \left[\sum [h_k]_e - b_k \sigma_k \right]^{1/2} \right\|_1 &\leq \left\| \left[\sum [h_k]_e - \langle h_k, \sigma_k \rangle \sigma_k \right]^{1/2} \right\|_1 \\ &\quad + \left\| \left[\sum |\langle h_k, \sigma_k \rangle - b_k|^2 \right]^{1/2} \right\|_1, \end{aligned}$$

which by Lemmas 4(II) and 5 is estimated by

$$C\|h - \sum \langle h_k, \sigma_k \rangle \sigma_k\|_1^{1/2} \|h\|_1^{1/2} + C\|h - f\|_1^{1/4} \|h\|_1^{3/4} \leq C\|h - f\|_1^{1/8} \|h\|_1^{7/8}.$$

Define for $u \in L^1(\Pi^N)$,

$$(u)_e(x) = \int_D u(\varepsilon_1 x_1, \varepsilon_2 x_2, \dots) d\varepsilon$$

(= the natural projection on the even part in $x_1, x_2, \dots$).

$$\text{LEMMA 7. } \|[\sum_k |(\hat{h}_k(1))_e|^2]^{1/2}\|_1 \leq C\|h - f\|_1^{1/16} \|h\|_1^{15/16}.$$

PROOF. At this point, we will make use of Proposition 7. Fix $x \in \Pi^N$ and remark that the sequence of functions in $\varepsilon \in D$,

$$[h_k]_0(\varepsilon_1 x_1, \varepsilon_2 x_2, \dots),$$

is a martingale difference sequence.

Moreover, the k th Rademacher coefficient is clearly given by

$$\sum_{n>0} (\hat{h}_k(n))_e(x) \sin nx_k$$

and Proposition 7 yields

$$\begin{aligned} & \left[\sum_k \left| \sum_{n>0} (\hat{h}_k(n))_e(x) \sin nx_k \right|^2 \right]^{1/2} \\ & \leq C \left[\int \left| \sum [h_k]_0(\varepsilon \cdot x) \right|^2 d\varepsilon \right]^{1/2} \left[\int \left[\sum |[h_k]_0(\varepsilon \cdot x)|^2 \right]^{1/2} d\varepsilon \right]^{1/2}. \end{aligned}$$

Integration in x , application of Cauchy-Schwarz and Lemma 6, gives

$$(+) \quad \left\| \left[\sum_k \left| \sum_{n>0} (\hat{h}_k(n))_e \sin nx_k \right|^2 \right]^{1/2} \right\|_1 \leq C\|h - f\|_1^{1/16} \|h\|_1^{7/16} \left\| \left[\sum |[h_k]_0|^2 \right]^{1/2} \right\|_1^{1/2}.$$

Also

$$\left\| \left[\sum |[h_k]_0|^2 \right]^{1/2} \right\|_1 \leq C\|h\|_1.$$

On the other hand, we can multiply the k th increment in the left member of (+) by $\sin x_k$ and then take $\mathbf{E}_{k-1}$ -expectation. Proposition 5 shows that

$$\left\| \left[\sum_k |(\hat{h}_k(1))_e|^2 \right]^{1/2} \right\|_1 \leq C\|h - f\|_1^{1/16} \|h\|_1^{15/16},$$

proving Lemma 7.

Now, rewriting

$$\left[\sum |[h_k]_e - b_k \sigma_k|^2 \right]^{1/2} = \left[\sum_k \left| \sum_{n>0} \hat{h}_k(n) \cos nx_k - b_k \sigma_k \right|^2 \right]^{1/2}$$

multiplication of the k th increment by $\cos x_k$ and taking $\mathbf{E}_{k-1}$ -expectation yields (by Proposition 5 and Lemma 6)

$$\left\| \left[\sum_k \left| \frac{1}{2} \hat{h}_k(1) - \frac{2}{\pi} b_k \right|^2 \right]^{1/2} \right\|_1 \leq C \|h - f\|_1^{1/8} \|h\|_1^{7/8}.$$

Since $b_k = (b_k)_e$, a convexity argument allows us to replace, in a previous inequality, $\hat{h}_k(1)$ by $(\hat{h}_k(1))_e$. Combining with Lemma 7, we conclude

$$C^{-1} \|f\|_1 \leq \left\| \left(\sum |b_k|^2 \right)^{1/2} \right\|_1 \leq C \|h - f\|_1^{1/16} \|h\|_1^{15/16}, \quad \|f\|_1 \leq C \|h - f\|_1,$$

and thus Proposition 10.

REFERENCES

1. A. Bonami, *Étude des coefficients de Fourier des fonctions de $L^p(G)$* , Ann. Inst. Fourier (Grenoble) **20** (1970), 335–402.
2. J. Bourgain, *On the embedding problem of L^1 in L^1/H^1* , Bull. Soc. Math. Belg. **34** (1982), 187–194.
3. J. Bourgain and H. P. Rosenthal, *Applications of the theory of semi-embeddings to Banach space theory*, preprint.
4. J. Bourgain, *New Banach space properties of the disc algebra and H^∞* , Acta Math. (to appear).
5. J. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
6. J. Garnett and P. Jones, *The distance in BMO to L^∞* , Ann. of Math. (2) **108** (1978), 373–393.
7. A. Garsia, *Martingale inequalities*, Math. Lecture Notes, Benjamin, New York, 1973.
8. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Vol. I, Ergeb. Math. Grenzgeb., no. 92, Springer-Verlag, Berlin and New York, 1977.
9. A. Pełczyński, *Banach spaces of analytic functions and absolutely summing operators*, CBMS Regional Conf. Ser. in Math., no. 30, Amer. Math. Soc., Providence, R.I., 1976.
10. E. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press., Princeton, N.J., 1970.
11. ———, *Classes HP , multiplicateurs et fonctions de Littlewood-Paley*, C.R. Acad. Sci. Paris Sér. A **263** (1966), 716–719; 780–781.

DEPARTMENT OF MATHEMATICS, VRIJE UNIVERSITEIT BRUSSEL, PLEINLAAN 2-F7, 1050 BRUSSELS, BELGIUM